Lecture 9: Lattice Spin Systems

$G = (V, E)$ finite, connected, a distinguished subset $B \subseteq V$ - "the boundary".

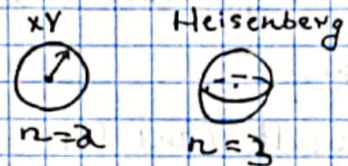
Spins take values in a state space (S, λ) which is a measure space (S - set with a σ-algebra, λ - measure.)

Examples: 1) Ising model

$S = \{+1, -1\}$, λ - counting

2) Spin $O(n)$ model, $n \geq 2$:

$S = S^{n-1}$, λ - Lebesgue measure



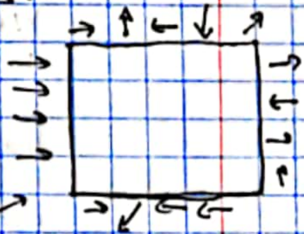
Given $\tau : B \to S$ boundary values, the measure on $\sigma : V \to S$, $\sigma|_B = \tau$ is: $\beta = \frac{1}{T}$, T = temp.

1) Ising model: $\mathbb{P}_G^\tau(\sigma) = \frac{1}{Z_G^\tau} e^{\beta \sum_{\substack{u \sim v \\ u, v \in V}} \sigma_u \sigma_v}$

2) Spin $O(n)$ models with $n \geq 2$, the density of σ w.r.t. the product Lebesgue measure is:

$$\frac{1}{Z_G^\tau} \cdot e^{\beta \sum_{\substack{u \sim v \\ u, v \in V}} \sigma_u \cdot \sigma_v}$$

Inner product in $\mathbb{R}^n$



a choice of τ for $n = 2$.

High-temperature regime

When β is low, there is no long-range order and correlations decay exponentially.

One instance of that is:

Thm. $G = \Lambda_L^d = \{-L+1, -L+2, \dots, L\}^d$

with $B = \{v \in \Lambda_L^d : \exists u \in \mathbb{Z}^d \setminus \Lambda_L^d, u \sim v\}$

Then for each $s \in S$ and $\tau \in S$,

$$\| E^{\tau}_{\Lambda_L^d} (\sigma(0,\dots,0)) \| \le C e^{-c \cdot L}$$

norm in $\mathbb{R}^n$

For the Ising and Spin $O(n)$ models when $\beta < \beta_0(n,d)$

To prove this we show a general, broadly applicable theorem, called the Dobrushin uniqueness thm.

Dobrushin's result (and later extensions) may be presented in the setting of the graph G, state space S and ~~bdry~~ bdry. Subset B as before. We requiere now S to be a Polish space (metric, seperabale and complete), in particular with a metric ρ. we also assume that λ is a measure on the borel σ-algebra of ρ.

We prove fast decay of correlations for measures μ on configurations $\sigma : V \to S$ having density w.r.t $\prod_{v \in V} \lambda(\sigma_v)$ with the density continuous and nowhere zero.

Boundary values: For each $V_0 \in V$ and $\tau : V_0 \to S$ we have the measure μ^τ which is μ conditioned on $\sigma|_{V_0} = \tau$. (using the cts. density and our assumption that it is nowhere zero)

Our goal is to show that in a suitable "low influence" and "nearest-neighbor" scenario, the bdy values do not affect μ^τ at positions far from the bdry.

We have two notions:

<u>Weak spatial mixing with constants $C, c > 0$</u>

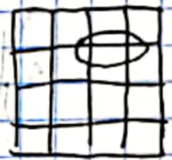
$\forall\, U \subseteq V,\ \forall\, \tau_1, \tau_2 : B \to S,$

$$d_W(\mu_U^{\tau_1}, \mu_U^{\tau_2}) \le C \cdot |U| \cdot e^{-c \cdot d_G(U,B)}$$

Wasserstein distance defined below

graph dist

μ_U^τ – the marginal on U of μ^τ.



<u>Strong spatial mixing with constants $c, C > 0$:</u>

$\forall\, U \subseteq V,\ \forall\, \tau_1, \tau_2 : B \to S,$

$$d_W(\mu_U^{\tau_1}, \mu_U^{\tau_2}) \le C\,|U|\, e^{-c\, d_G(U,\, B_{\tau_1,\tau_2})}$$

$B_{\tau_1,\tau_2} = \{ v \in B : \tau_1(v) \neq \tau_2(v) \}$

<u>Distance on measures d_W</u>

$n = 2$
ρ = Euclidean distance.

Assume ρ has a <u>bounded diameter</u>, i.e. $\sup_{x,y \in S} \rho(x,y) < \infty$

d_W is the Wasserstein $(p = 1)$ distance:

For ν_1, ν_2 – measures on S,

$$d_W(\nu_1, \nu_2) := \min E(\rho(X,Y))$$

coupling of ν_1, ν_2 $\left\{ \begin{array}{l} (X,Y) \text{ RVs in } S \\ X \sim \nu_1 \\ Y \sim \nu_2 \end{array} \right.$

An optimal coupling always exists by compactness (the min is achieved).

Ex. ρ - Hamming dist. : $\rho(x,y) = \mathbb{1}_{x \neq y}$

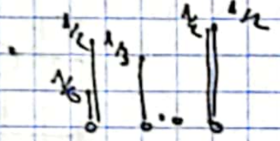
then d_W = total variation distance.

E.g $S = \{1,2,3\}$

$$\nu_1 = \begin{cases} 1 & 1/6 \\ 2 & 1/3 \\ 3 & 1/2 \end{cases} \qquad \nu_2 \sim \begin{cases} 1 & 1/2 \\ 2 & 0 \\ 3 & 1/2 \end{cases}$$

An optimal coupling -
$$\begin{array}{c|ccc|c} & 1 & 2 & 3 & \\ \hline 1 & 1/6 & 1/3 & 0 & 1/2 \\ 2 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1/2 & 1/2 \\ \hline & 1/6 & 1/3 & 1/2 & \end{array}$$

$$d_W(\nu_1,\nu_2) = \mathbb{E}\left(\mathbb{1}_{x \neq y}\right) = \mathbb{P}(X \neq Y) = \frac{1}{3}.$$



When ν_1, ν_2 are measures on S^k we take:

$$d_W(\nu_1,\nu_2) := \min_{\substack{(X,Y) \text{ RVs in } S^k \\ X \sim \nu_1 \\ Y \sim \nu_2}} \mathbb{E}\left[\sum_{j=1}^{k} \rho(X_j, Y_j)\right].$$

Fact: d_W satisfies the triangle inequality.

Low influence condition

Define, for $u,v \in V$, $u \neq v$,

$I_{u \to v}$ as the minimal non-negative real s.t for all $\tau_1, \tau_2 : V \setminus \{v\} \to S$, $\tau_1 = \tau_2$ everywhere except on u.

$$d_W\left(\mu_v^{\tau_1}, \mu_v^{\tau_2}\right) \leq I_{u \to v}\, \rho\left(\tau_1(u), \tau_2(u)\right).$$

- We say that μ is nearest neighbor if $I_{u \to v} = 0$ whenever $u \nsim v$.

Thm. (Dobrushin Uniqueness
Dobrushin 1968 and later extensions):

Suppose μ is nearest neighbor. Let $\alpha = \max_{v \in V} \sum_{u \in V \setminus \{v\}} I_{u \to v}$

If $\alpha < 1$ (Dobrushin condition), then for each $\tau_1, \tau_2 : B \to S$, there exists a coupling of $\mu^{\tau_1}, \mu^{\tau_2}$ s.t for each $v \in V$, $\mathbb{E}\left[\rho(\sigma_1(v), \sigma_2(v))\right] \leq \mathrm{diam}(\rho) \cdot \alpha^{d_G(v, B_{\tau_1,\tau_2})}$

(Where (σ_1, σ_2) are sampled from the coupling).

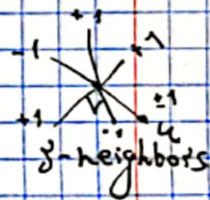
In particular, we have strong spatial mixing:

$$d_W(\mu_U^{\tau_1}, \mu_U^{\tau_2}) \leq \mathrm{diam}(\rho) \cdot |U| \cdot \alpha^{d_G(U, B_{\tau_1,\tau_2})}$$

__Example for Ising model__: $S = \{1, -1\}$, ρ = Hamming metric ($\rho(x,y) = \mathbb{1}_{x \neq y}$).

Take G to be a graph with maximal degree Δ.
$\exists$ a universal $C > 0$ s.t Dobrushin's cond. holds when $\beta < \frac{C}{\Delta}$ (on $\mathbb{Z}^d$, $\beta < \frac{C}{2d}$).

To see this, let v be a vertex of degree δ.



Fix bdry conditions τ_1, τ_2 on $V \setminus \{v\}$ which are equal except at u. Suppose:

$$|\{z : z \sim v, \tau_1(z) = +1\}| = k$$
$$|\{z : z \sim v, \tau_2(z) = +1\}| = k+1$$

Then $\mu_v^{\tau_1}(+1) = \dfrac{e^{\beta(2k-\delta)}}{e^{\beta(2k-\delta)} + e^{-\beta(2k-\delta)}} = p_k$.

$$d_W(\mu_v^{\tau_1}, \mu_v^{\tau_2}) = \mu_v^{\tau_2}(+1) - \mu_v^{\tau_1}(+1) = d_{TV}(\mu_v^{\tau_1}, \mu_v^{\tau_2})$$

$$I_{u \to v} = \max_{0 \leq k \leq \delta - 1} (p_{k+1} - p_k) \underset{\substack{\uparrow \\ \text{max is achieved} \\ \text{at } k=0}}{=} p_1 - p_0 = \frac{1 - e^{-4\beta}}{\left(e^{2\beta(2-\delta)} + 1\right)\left(e^{-2\beta\delta} + e^{-4\beta}\right)}$$

Dobrushin cond. turns out to be:

$$\Delta \cdot \frac{1 - e^{-4\beta}}{\left(e^{2\beta(2-\Delta)} + 1\right)\left(e^{-4\beta} + e^{-2\beta\Delta}\right)} < 1.$$

Remark: Influence matrix $\to$ $I = v\begin{pmatrix} & u & \\ \cdots & I_{u\to v} & \end{pmatrix}$ $\quad d = \|I\|_{\ell_\infty \to \ell_\infty}$.

There are $\underbrace{\text{versions}}_{\text{of Dobrushin's result}}$ with other norms, of I.

proof: (of thm.)

Preliminary lemma: For any $v \in V$ and $\tau_1, \tau_2 : V\setminus\{v\} \to S$

$$d_W(\mu_v^{\tau_1}, \mu_v^{\tau_2}) \leq \sum_{u \in V\setminus\{v\}} I_{u\to v}\, \rho(\tau_1(u), \tau_2(u))$$

(i.e, the def. of influence extends ~~additively~~ additively to the case that τ_1, τ_2 may differ at many vertices).

proof: Interpolate τ_1 and τ_2:

Fix some ordering $(u_n)_{n=1}^{|V|-1}$ of $V\setminus\{v\}$,

$$\tau^m(u_n) = \begin{cases} \tau_1(u_n) & n > m \\ \tau_2(u_n) & n \leq m \end{cases}$$

So that $\tau^0 = \tau_1$, $\tau^{|V|-1} = \tau_2$. By the def. of influence:

$$d_W(\mu_v^{\tau_1}, \mu_v^{\tau_2}) \underset{\substack{\uparrow \\ \text{triangle ineq.}}}{\leq} \sum_{m=1}^{|V|-1} d_W(\mu_v^{\tau^{m-1}}, \mu_v^{\tau^m}) \leq$$

$$\leq \sum_{m=1}^{|V|-1} I_{u_m \to v}\, \rho(\tau_1(u_m), \tau_2(u_m)).$$

We now prove the existence of the coupling.

We wil define a sequence of couplings.

~~For each $n \geq 0$, we define a seq. of couplings~~.

For each $n \geq 0$, we define (σ_n, σ_n') s.t

1) $\sigma_n \sim \mu^{\tau_1}$, $\sigma_n' \sim \mu^{\tau_2}$

2) For each $v \in V$,

$$\mathbb{E}[\rho(\sigma_n(v), \sigma_n'(v))] \leq \operatorname{diam}(\rho)\, d^{\min\{n,\, d_G(v, B_{\tau_1,\tau_2})\}}$$

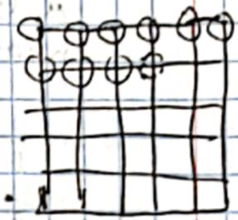
The required coupling is the case when $n \geq$ diameter of G.

The couplings are found by induction.

$\underline{n=0}$: Let σ_n, σ_n' be independent with $\sigma_n \sim \mu^{\tau_1}, \sigma_n' \sim \mu^{\tau_2}$.

Induction step: Assume we have the coupling $(\sigma_{n-1}, \sigma_{n-1}')$.

To define (σ_n, σ_n'), let $(v_k)_{k=1}^{|V \setminus B|}$ be an arbitrary ordering on $V \setminus B$.



Define $(\sigma_{n,k}, \sigma_{n,k}')_{k=0}^{|V \setminus B|}$ inductively on k as follows.

we have inductions both on n and on k. $(\sigma_{n,k}, \sigma_{n,k}')$ is obtained from $(\sigma_{n,k-1}, \sigma_{n,k-1}')$ by

$\underline{k=0}$: $\sigma_{n,0} = \sigma_n$, $\sigma_{n,0}' = \sigma_n'$.

For each $k \geq 1$, resampling the value of $[\sigma_{n,k-1}, \sigma_{n,k-1}']$ at v_k from the ~~original~~ optimal coupling of the marginal dist. there given all values of $(\sigma_{n,k-1}, \sigma_{n,k-1}')$ at the other vertices.

Finally, set $\sigma_n = \sigma_{n,|V \setminus B|}$, $\sigma_n' = \sigma_{n,|V \setminus B|}'$.

With this procedure, for each n,k, $\sigma_{n,k} \sim \mu^{\tau_1}, \sigma_{n,k}' \sim \mu^{\tau_2}$.

In addition, we now show that:

$$\mathbb{E}[\rho(\sigma_{n,k}(v_k), \sigma_{n,k}'(v_k))] \leq \mathrm{diam}(\rho) \cdot \alpha^{\min(n, d_G(v_k, B_{\tau_1,\tau_2}))}$$

To see this, let's use the preliminary lemma to calculate what the optimal coupling gives.

$$\mathbb{E}[\rho(\sigma_{n,k}(v_k), \sigma_{n,k}'(v_k)) \mid \sigma_{n,k-1}, \sigma_{n,k-1}'] \leq$$

By the choice of optimal coupling, this is d_W of the marginal dist. at ~~given~~ v_k given $\sigma_{n,k-1}, \sigma_{n,k-1}'$ at all other vertices

$$\overset{\text{by lemma}}{\leq} \sum_{u \in V \setminus \{v_k\}} I_{u \to v_k} \rho(\sigma_{n,k-1}(u), \sigma_{n,k-1}'(u))$$

$$\Rightarrow \mathbb{E}[\rho(\sigma_{n,k}(v_k), \sigma_{n,k}'(v_k))] \leq \sum_{u \in V \setminus \{v_k\}} I_{u \to v_k} \underbrace{\mathbb{E}[\rho(\sigma_{n,k-1}(u), \sigma_{n,k-1}'(u))]}$$

$$\underset{\text{by the inductions}}{\leq} \mathrm{diam}(\rho) \begin{cases} \alpha^{\min\{n-1, d_G(u, B_{\tau_1,\tau_2})\}} & m \geq k, u = u_m \\ \alpha^{\min\{n, d_G(u, B_{\tau_1,\tau_2})\}} & m < k, u = u_m \end{cases}$$

$$\overset{\mathrm{diam}(\rho)}{\le} r \sum_{u:\, u\sim v_u} I_{u\to v_u} \cdot \alpha^{\min\{h-1,\, d_{G_0}(u, B_{\tau_1,\tau_2})\}}$$

u is nearest neighbor

$$\overset{\mathrm{diam}(\rho)}{\le} \sum_{u:\, u\sim v_u} I_{u\to v_k}\, \alpha^{\min\{h-1,\, d_G(v_k, B_{\tau_1,\tau_2})\}}$$

$$\overset{\mathrm{diam}(\rho)}{=} r\, \alpha^{\min\{h,\, d_G(v_k, B_{\tau_1,\tau_2})\}-1} \underbrace{\sum_{u:\, u\sim v_u} I_{u\to v_u}}_{\le 1}$$

$$\overset{\mathrm{diam}(\rho)}{\le} r\, \alpha^{\min\{h,\, d_G(v_k, B_{\tau_1,\tau_2})\}}$$

$\square$